Categorical Harmony and Path Induction

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June 9, 2016

Outline

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1. Introduction
2. Inferentialism
3. Harmony
   - Inversion
   - Inferential
   - Conservative
   - Categorical
4. Justifying Path Induction

Motivation

- Homotopy Type Theory (HoTT) is a novel connection between classical homotopy theory and intuitionistic, intensional type theory.

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- It can be presented formally with rules for type formation, term introduction, term elimination, and computation.

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- This rule differs from normal reasoning about identity; it is not merely Leibniz's Law.

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$$\frac{x : A, y : A, p : \mathtt{Id}_A(x, y) \vdash C(x, y, p)\,\mathtt{Type} \qquad x : A \vdash t(x) : C(x, x, \mathtt{refl}_x)}{x : A, y : A, p : \mathtt{Id}_A(x, y) \vdash J(t; x, y, p) : C(x, y, p)}$$

where A is a type.

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where A is a type.

- This allows us to construct a term J of a type C dependent on an identification of two terms of A as long as we've given a term of that type for the trivial identification.

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- After all, if HoTT is supposed to serve as a foundation, using homotopy theory to justify a rule of inference seems backwards.

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- They give such a justification, that breaks path induction into two principles whose combination is equivalent to path induction.
- We'll come back to this.

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- I want to give an alternate justification that gives up the pre-mathematical requirement.
- Instead, I give a mathematically motivated justification of path induction.

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- The inferential role of a connective or concept completely characterizes that connective or concept.

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- Now we have to find an elimination rule that appropriately 'balances' this introduction rule, but needs no other justification.
- We check whether candidate elimination rules are correct through a notion of harmony.
- There are currently many conceptions of harmony. *I want to unify these conceptions of harmony.*

Making inferentialism precise

- We use category theory as a coherent way to talk about inferentialism, to specify when a concept is *'meaning-bearing'* or *harmonious*.

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- Why category theory?
    - Distinguished objects are constructed in an inferential way, from the structure of arrows e.g. limits and colimits.
    - Yoneda Lemma: Interpret arrows into an object A as 'generalized objects' of A and considering all such generalized objects of A uniquely characterizes A.

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Whence Harmony?

- The original motivation of harmony came in response to Arthur Prior's connective *tonk* with the following rules of inference:

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3 Conceptions

1. Inversion Harmony
2. Inferential Harmony
3. Conservative Harmony

Inversion Principle

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- This is the inversion principle that Prawitz uses to define harmony.
- This sort of harmony is witnessed by a *local reduction*.

Example: Conditional

- We can reduce derivations of the form

$$\dfrac{\dfrac{\begin{matrix}[A] \\ \vdots \\ B\end{matrix}}{A \to B}\,\to\mathsf{I} \qquad \begin{matrix}\mathcal{D} \\ A\end{matrix}}{B}\,\to\mathsf{E}$$

to the derivation

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- We remove the 'detour' of $\to$I followed by $\to$E.

Stability

- Dual to the notion of inversion harmony, we have local expansions that give us *stability* such as the following

$$\begin{array}{c}\mathcal{D}\\ A \to B\end{array} \quad \text{to proofs of the form} \quad \dfrac{\dfrac{\begin{array}{c}\mathcal{D}\\ A \to B\end{array} \qquad [A]^u}{B}\,{\to}\mathsf{E}}{A \to B}\,{\to}\mathsf{I},\ \mathsf{u} \quad .$$

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- We will capture both inversion harmony and stability in categorical terms.

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- Strength: $A\lambda B$ is the strongest conclusion possible under conditions described by λ-I. Moreover, in order to show this
  i. one needs to exploit all the conditions described by λ-I;
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- Weakness: $A\lambda B$ is the weakest major premise under the conditions described by λ-E. Moreover, in order to show this
  i. one needs to exploit all the conditions described by λ-E;
  ii. one needs to make full use of λ-I; but
  iii. one may not make any use of λ-E.

Example: Conditional

- Strength: $A \to B$ is the strongest proposition implied by Γ whenever $\Gamma \cup \{A\}$ entails B.

Proof.

Suppose there is a θ that is implied by Γ whenever $\Gamma \cup \{A\}$ implies B. By $\to$E, we know that $A \to B$ fits the role of Γ, giving us the desired

$$A \to B \vdash \theta$$

$\square$

Example: Conditional

- Weakness: $A \to B$ is the weakest proposition that with A implies B.

Proof.

Suppose θ is a proposition that with A implies B. By $\to$I, we get

$$\dfrac{\dfrac{\theta \qquad [A]^u}{B}}{A \to B}\,{\to}\mathrm{I,u} \quad ,$$

which shows that θ implies $A \to B$ as desired. □

Conservative Extension

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- So for a system S that includes the introduction and elimination rules (λ-I, λ-E) for a connective λ, these rules are harmonious if S is a conservative extension of $S \backslash (\lambda\text{-I}, \lambda\text{-E})$.

Uniqueness

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- If the connective λ has rules of inference R, then we say R ensures the uniqueness of λ when for any new connective λ', we can show that when we replace λ with λ' in all rules of R, we can derive

$$\lambda(x_1, \ldots, x_n) \leftrightarrow \lambda'(x_1, \ldots, x_n).$$

Uniqueness Example

- Claim: The rules for conjunction ensure the uniqueness of $\wedge$.

Proof.

Suppose that there is a connective $\wedge^*$ with the following rules of inference:

$$\frac{A \qquad B}{A \wedge^* B}\,\wedge^*\mathrm{I}$$

$$\frac{A \wedge^* B}{A}\,\wedge^*\mathrm{E} \qquad \frac{A \wedge^* B}{B}\,\wedge^*\mathrm{E}$$

Given these rules, which are just the rules for conjunction with $\wedge^*$ replacing $\wedge$, we can show the necessary biconditional via the following derivations:

$$\frac{\dfrac{A \wedge B}{A} \qquad \dfrac{A \wedge B}{B}}{A \wedge^* B}\,\wedge^*\mathrm{I} \qquad \frac{\dfrac{A \wedge^* B}{A} \qquad \dfrac{A \wedge^* B}{B}}{A \wedge B}\,\wedge\mathrm{I}$$

□

Adjoints

- We unify the important parts of the above conceptions into one, using the categorical notion of an *adjoint*.

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Claim

Any operation (connective, concept) produced by adjunction from universally definable operations is harmonious.

Adjoints

Definition

Let $L : \mathbb{C} \to \mathbb{D}$ and $R : \mathbb{D} \to \mathbb{C}$ be functors. L is *left adjoint* to R, or R is *right adjoint* to L, denoted $L \dashv R$, if the following holds:

Natural isomorphism: There exists a natural isomorphism

$$\theta : \mathrm{Hom}_{\mathbb{D}}(L(-), -) \simeq \mathrm{Hom}_{\mathbb{C}}(-, R(-))$$

which is to say that for any objects $C \in \mathbb{C}$ and $D \in \mathbb{D}$, we have a bijection between the hom sets

$$\frac{\mathrm{Hom}_{\mathbb{D}}(L(C), D)}{\mathrm{Hom}_{\mathbb{C}}(C, R(D))}\;\theta(C, D)$$

that is natural in C and D.

Universally Definable Operations

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- Example: Terminal $\mathbf{1} : \mathbb{C} \to \mathbb{1}$ that takes objects to the unique element $*$ of $\mathbb{1}$ and arrows to the identity on $*$.

Harmonious Operations

- From Lawvere's work, we get that the following connectives are categorically harmonious: $\wedge, \vee, \top$, and $\bot$ since

$$\vee \dashv \mathbf{\Delta} \vdash \wedge$$

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- In Lawvere's hyperdoctrines, we also get that the quantifiers $\forall$ and $\exists$ are harmonious.

Inversion in Categorical

- Let L be a set of well-formed formulae that is closed under some inductive term constructors and let $\vdash_L$ be a preorder on L.

Definition

[a] A category of provability $(L, \vdash_L) = \mathbb{L}$ is defined by

- The objects are $A, B, C, \ldots \in L$;
- The arrows are derivability claims: there is an arrow $p : A \to B$ if and only if $A \vdash_L B$.

[a] This categorical proof theory is taken from Ed Morehouse's dissertation, where he shows inversion harmony in $(L, \vdash_L)$.

Example: Conjunction

- Define conjunction as the right adjoint to the diagonal:

$$\Delta \dashv \wedge\,.$$

This gives us the biconditional

$$\frac{(A, A) \to (B, C)}{A \to B \wedge C}$$

which amounts to

$$\frac{A \;\vdash_L\; B \qquad A \;\vdash_L\; C}{A \;\vdash_L\; B \wedge C}$$

which is a sequent description of $\wedge$I.

Example: Conjunction

- For $\wedge$E, we look to the counit. The counit gives us the biconditional

$$\frac{A \wedge B \ \vdash_L \ A \qquad A \wedge B \ \vdash_L \ B}{A \wedge B \ \vdash_L \ A \wedge B} \ .$$

  And since $\vdash_L$ is reflexive, we have an arrow on the bottom, giving us both elimination rules.

Example: Conjunction

- We have local reduction by the universal property of the counit. That is, two proofs of the form

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}}{A \wedge B}}{A} \quad \text{and} \quad \dfrac{\dfrac{\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \qquad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix}}{A \wedge B}}{B}$$

are uniquely associated to two proofs of the form

$$\begin{matrix}\mathcal{D}_1 \\ A\end{matrix} \quad \text{and} \quad \begin{matrix}\mathcal{D}_2 \\ B\end{matrix} \; .$$

proof

Unifying Harmony

- Categorical Harmony captures
  - Inversion harmony (local reduction, local expansion, permutation).
  - Strength and Weakness conditions of inferential harmony.
  - Uniqueness of connectives from conservative harmony.

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  - Restrictions on how you prove inferential harmony (e.g. full use of λ-E).
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- Categorical Harmony **does not** capture
  - Restrictions on how you prove inferential harmony (e.g. full use of λ-E).
  - Conservative extensions. See Adam Simon's talk tomorrow!
- Note: Tonk is not definable by iterated adjoints on universally definable operations.

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- In hyperdoctrines, identity is defined by an adjunction from universally definable operations and thus categorically harmonious.
- In proof-relevant hyperdoctrines with a comprehension functor, path induction comes from this adjunction, just like $\wedge$I and $\wedge$E came from the adjunction for conjunction. statement of theorem
- Thus path induction comes from the harmonious definition of identity in HoTT, showing that path induction is justified.

Challenge Reconsidered

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  - *Uniqueness principle for identity types*: All proofs of identity are propositionally equal to the trivial self-identification.
  - *Substitution salva veritate*: If $P(a)$ and $a = b$, then $P(b)$.
- These principles are equivalent to *contractibility* and *transport* in the HoTT literature and their combination is equivalent to path induction.

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- We accept that *substitution salva veritate* is part of our pre-mathematical understanding of identity.
- We do not accept that the *uniqueness principle for identity types* is pre-mathematical and is in fact counter to pre-theoretic ideas.

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- *Uniqueness of identity proofs*:

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- Uniqueness of identity proofs makes HoTT extensional! Contradicts Univalence!

Conclusion

- Instead of the pre-mathematical requirement, we gave a way to see that path induction is the correct elimination rule based on principles underlying the type theory, i.e. harmony.

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- Instead of the pre-mathematical requirement, we gave a way to see that path induction is the correct elimination rule based on principles underlying the type theory, i.e. harmony.
- We used category theory to unify harmony and to codify an inferentialist strategy of justification that showed that path induction is the morally correct identity elimination rule.

Thank You!

Inversion from adjunction

Proof.

To see this, we take $\mathcal{D}_1 : \Gamma \to A$ and $\mathcal{D}_2 : \Gamma \to B$ creating an arrow $(\mathcal{D}_1, \mathcal{D}_2) : \Delta(\Gamma) \to (A, B)$. By the universal property of the counit, this gives us a unique arrow $f : \Gamma \to A \wedge B$ such that

$$(\mathcal{D}_1, \mathcal{D}_2) = \varepsilon_{(A,B)} \circ \Delta(f).$$

And since the counit is given by the pair of 'projections' $(p_1, p_2) : A \wedge B \to (A, B)$ and $f = \langle \mathcal{D}_1, \mathcal{D}_2 \rangle$, we get

$$(\mathcal{D}_1, \mathcal{D}_2) = (p_1, p_2) \circ \Delta\langle \mathcal{D}_1, \mathcal{D}_2 \rangle,$$

giving us the local reduction. □

Go Back

Precise statement of theorem

- Let $(\mathbb{C}, \mathcal{P})$ be a hyperdoctrine with the comprehension functor R which is right adjoint to the functor

$$L: \ \mathbb{C}/X \to \mathcal{P}(X)$$

defined on an arrow $\alpha : A \to X$ by $L(\alpha) := \exists_\alpha \top_A \quad \in \mathcal{P}(A)$.

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- Path induction is admissible in hyperdoctrines that satisfy the following two conditions:
  1. $R \circ L = F$ is idempotent, i.e. $F^2 \cong F$.
  2. For $\mathcal{F}$ the full subcategory that is the image of R, if $\alpha : A \to X \in \mathcal{F}_X$ and $f : X \to Y \in \mathcal{F}_Y$, then $f \circ \alpha : A \to Y \in \mathcal{F}_Y$.

Go Back