The Right Level of Abstraction: Category theory and methodological frames

Patrick Walsh

Carnegie Mellon University

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Outline

1. Reformulating Foundations

2. Category Theoretic Tools

Conclusions

Claim 1

Foundational debates are better served with a focus on the relationships between methodological perspectives, not just picking your favorite.

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Claim 2

Category theory affords a way to find the correct level of abstraction to answer some foundationally important questions.

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- But this can often be framed as a battle of dogmas, trained intuitions, and sometimes even social or political power.
- I would like to focus instead on what Wilfried Sieg calls "methodological frames".

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  - Like the deep commonality between intuitionistic and classical logic.
- I want to characterize the views on what objects should be taken as mathematically basic without subscribing to any particular one.
  - The Natural Numbers $\mathbb{N}$
  - Recursive ordinals $\mathcal{O}$
  - Segments of the Cumulative Hierarchy V

    $\vdots$

Methodological Frames

Punchline

The methodological frames are not incommensurable or all that dissimilar; they are related in deep and philosophically interesting ways.

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Thus, the frames are what Sieg calls *accessible domains*.

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Non-examples:

- The (classical) real numbers
- Class of theorems of a formal language

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Why is category theory an appropriate tool?

1. Give a good ‘third-order’ perspective.
2. Focuses on the relations between objects and not their internal structures.

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2. The second-order perspective looks at first-order objects arranged into structures: the *ordered* set of natural numbers, a topological space, methodological frames.
3. The third-order perspective looks at the relations and properties of the second-order structures.

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Category theory finds the third-order perspective natural and easy.

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- Categories are defined by a class of 'objects' and 'arrows' that have to satisfy some basic laws, familiar from algebra.
- This allows us to treat very complicated structures as 'objects' as long as they satisfy the rest of the category axioms.
- The 'objects' in a category have only the properties that are expressible in terms of the 'arrows' of the category.

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- The characterization of accessible domains creates a single dimension, functors, with which to compare them

  Much like Klein's program gives a way to uniformly compare geometries via transformation groups.

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Different endofunctors give different accessible domains:

- The functor $F(X) = 1 + X$ gives rise to $\mathbb{N}$
- The functor $F(X) = 1 + X + X^{\mathbb{N}}$ gives rise to the second-number class.
- The von-Neumann universe V comes from the 'powerclass' functor.

Conclusion

I have argued for:

1. Pluralistic inquiries into various methodological frameworks.
2. Category theory is a useful tool when a third-order perspective is needed.

Thanks!