Structuralism: A Defense

A first defense of the metaphysics of mathematical structuralism

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Abstract: I first establish the proper relationship between philosophy and mathematics in order to suppose that professional mathematicians have some idea of what they do while still maintaining philosophy's ability to prescribe what mathematics is. With a firm grounding in where philosophy stands with respect to mathematics, I begin to defend a normative framework for modern mathematics. I compare the axiom systems of set, group, and category theory to establish this framework. The main distinction to notice between the axioms is that of strong and weak axioms. I argue that the strong axioms are inessential to a mathematical ontology and the weak axioms preserve truth and provability while asserting a modest ontology. I use this conclusion to support structuralism. I then provide an empirical argument for structuralism based on the unreasonable effectiveness of mathematics in the natural sciences. I then argue for the abstract existence of such structures. In the end, I hope to have defended structuralism.

Introduction:

In an undergraduate education in mathematics, it is rare to investigate the philosophical foundations upon which the field rests. Even mathematical logic can be unreflectively axiomatic. This lack of critical reflection in the vast majority of practicing mathematicians provides a great many questions that the philosophy of mathematics attempts to answer. It is often teased that the mathematician is a Platonist during the week and a constructivist on the weekends. This is simply because the mathematician must believe they are discovering truths about existing objects, not invented phantasms. But on the weekend, in discussions with friends, they cannot seem to defend such a belief so they revert to a grudging constructivism. Obviously, this does not imply that mathematics *is* about abstract objects through the week and then becomes a human invention on Saturday and Sunday. So the philosophy of mathematics attempts to find a framework of mathematics that permits the intuition of mathematicians while introducing the rigor needed to satisfy the philosopher.

In this paper, I defend the idea that mathematics is discovered by showing how realism is the only way to incorporate the unreasonable effectiveness of mathematics. The unreasonable effectiveness of mathematics is taken to mean the unexpected usefulness of mathematics in explaining the natural world. If mathematics is arrived at a priori, then how exactly does it pertain so perfectly and universally to the physical world? But that is only the first step. We must first define the realism to be defended. I advocate for structural realism. This position asserts that mathematics is discovered and independent of minds. But the object of mathematics is not the number or line; it is the structures containing numbers, lines, operations and the like. In other words, mathematics does not discover things about first-order objects, only higher order objects. This is supported by the axiom systems, mathematical intuition, and empirical facts. The axiom systems of group theory, category theory, and especially the disparity between two set theories imply that mathematics would be inconsistent if it was about first-order objects. The mathematical intuition exemplified in the idea of morphism provides an argument for structure-based inquiry in mathematics. Finally, the usefulness of mathematics in the natural sciences supports a realist and structural approach.

Before I defend structural realism, I must first support the way I go about that defense. We must know from where to start our discussion. So I begin by defending something called the parallel philosophy stance in relation to professional mathematicians. This will allow me to use facts about mathematics currently held to support my thesis of structuralism.

Philosophy and Mathematics:

Philosophy stands as foundation and often progenitor of many disciplines and academic fields. The relation between philosophy and these others is often seen as a mixture of positive and normative claims. Not only does the philosopher of science describe what he thinks science *is* doing, he also prescribes the line between science and non-science, as Karl Popper has attempted. I claim that the philosophy of mathematics should be neither philosophy-first nor philosophy-last. It is a crucial assumption that many philosophers use in order to do both analytic description and normative prescription. I will make plain why there should be a sort of give and take between mathematics and philosophy.

The first stance on the relation between mathematics and philosophy that we shall consider is what I'll call the philosophy-first position. Philosophy-first asserts that philosophy should not be sensitive to what the professional mathematicians think they do. We shouldn't, in other words, ask the mathematician why proof confers mathematical truth or how logic plays into mathematics. Instead, philosophy largely goes over the heads of mathematicians in order to describe the field. One immediate consequence of this is the possibility that mathematicians are not doing math at all. For example, if the philosophy-first thinkers decide that mathematics is really based on empirical observation only and is in fact not an abstraction at all, then mathematicians are all doing something that is evidently not math.

Not only would this put an odd linguistic strain on both parties (using "mathematics" to refer to two mutually exclusive practices), it would be able to suppose nothing of mathematical truth. So even though mathematicians hold that pi is irrational, the philosophy-first philosophers would not be able to suppose its truth unless derived from their purely philosophically based mathematics. In effect, this implies that the philosophy-first philosopher is not a philosopher of mathematics at all. It is simply a pun to call them so, as they could be using the word "mathematics" as something that doesn't pertain at all to what anyone else calls mathematics. Barring the discussion in philosophy of language on this issue, it suffices to say that the philosophy-first stance, although possibly consistent, cannot participate in the philosophy of mathematics.

Naturally, the next stance that also provides no answers to the question of mathematics is the philosophy-last stance. I take this stance to mean that the philosopher should respect all the assertions of mathematicians as true of mathematics, given good reasoning and such. This can, but doesn’t have to, imply that the philosophy of mathematics relies on the agreement of mathematicians. This can have dire consequences if we view the truth of mathematical claims being derived from the agreement of mathematicians. Later, I will show what these consequences might be.

The philosophy-last philosopher will be committed to a mathematician’s ontology and epistemology, which is admittedly unreflective. In other words, the philosopher would be making the exact claims a mathematician would but under the guise of philosophy. The philosophy-last stance implies a sort of lame philosophy of mathematics that simply states facts about mathematics with virtually no normative claims unique to a philosophy. Whereas the philosophy-first stance implies the philosophy is entirely prescriptive, the philosophy-last stance implies the philosophy is entirely descriptive. Neither seems apt to handle any of the questions of mathematics and should be put aside in favor of the stance that I support: the philosophy of mathematics should be sensitive to changes and developments in professional mathematics while still asserting norms by which the progress of mathematics should be conditioned. I will call this position parallel philosophy.

The Axioms of Mathematics:

The axioms in mathematics are often thought of as obviously true, true by definition, or as some sort of self-evident common notion. For illustration purposes we will use set theory, category theory, and a bit of group theory to explore the behavior of axioms. Professional set theorists tend to accept a modified version of Zermelo – Frankel set theory to be the standard axiomatization of sets. We will explore some of the specific axioms later but for now we will look at the basic concepts required to formulate the axioms. The notions of set and set membership are left undefined in Zermelo-Frankel set theory with the axiom of choice (ZFC). Thus we usually try to hone the definition implicitly through what might be called the "dialectical approach"[1]. This means that we give a first approximation of the basic notions and refine them as we encounter problems with that approximation. In other words, we define the concepts rather negatively, showing what they are not.

The first concept that needs some naïve description is the concept of a set. People usually have a good idea of what a set is like. For now, let's think of sets as definable collections of some objects, mathematical or otherwise. But in the extrema of this notion, sets start to exhibit strange behaviors that force us to refine our notion. The first problem we encounter is Russell's Paradox. Bertrand Russell discovered that naïve set theory (unaxiomatic intuitive theory) created a contradiction. The contradiction is simple: Let R be the set of all sets that are not members of themselves. Since set membership seems bivalent, either R is a member of itself or it is not. If R is a member of itself, then by definition of the members of R, R is not a member of itself. If R is not a member of itself, then it fits the definition of a member of R, so it is a member of itself. Thus we have a paradox. So our definition of sets as definable collections has already led to a serious problem.

This is in direct contrast to Gottlieb Frege's Basic Law V. This law states:

[1] Shapiro (1997)

> The course-of-values of the concept *f* is identical to the course-of-values of the concept *g* if and only if *f* and *g* agree on the value of every argument (i.e., if and only if for every object *x*, $f(x) = g(x)$) [2]

Russell's paradox puts limits on what counts as a set and what does not. The set R above does not count as a set because it implies a contradiction. So the axiom schema of comprehension[3] was introduced in ZFC to only allow one to take subsets of sets you already had. You could not create larger sets arbitrarily (there are axioms that allow you to 'grow' sets from others, but not in the least bit arbitrarily), only smaller ones. The idea of size here is informal. The set {1,2,3} is smaller and a subset of {1,2,3,4}. Of course, we aren't limited to finite sets, but the intuition will suffice for now. Technically, the axiom schema of comprehension is not one axiom but a recipe for infinitely more axioms. This is useful in avoiding any unsavory participation in Gödel's incompleteness theorems. The second axiom used to respond to Russell's Paradox is the foundation axiom[4]. This axiom implies that no set is a member of itself. This is a direct fix to the paradox and allows us to further refine our notion of set, but the axiom exists almost only to fix the paradox, so it has limited use.

We might wonder if the whole endeavor of set theory is to define, implicitly, the notion of set and then the notion of set membership. All the theorems and literature on set theory refines the idea of set, even a little bit. So we might think that our common notions are finally defined after we "finish" the theory of sets. However, this is viciously circular. In our axioms, we assert the existence of sets. We cannot hope to define sets by the theorems that we derive from these very same axioms. At least, if we do, it does not suffice to provide a real definition of our

[2] Frege (1964)
[3] See Appendix A.
[4] See Appendix A.

common notions. Most would argue that to define the common notions in terms of other notions would be contradicting the very notion of self-evidence or fundamental concept. So we see that even in a ‘complete’ theory of sets, we are no closer to finding out what a set really is at ground level.

What about a different axiomatization? Let’s look at a helpfully different axiom system of set theory: Von Neumann – Bernays – Gödel set theory (NBG). We needn’t go over the axioms in detail to make the distinction clear. As set theorists often boast, set theory is a powerful enough mathematical language to express most of the mathematics that does not deal directly with sets. Take arithmetic, for example. Both ZFC and NBG express all of arithmetic in terms of sets. It even allows us to use ordinals ‘larger’ than the infinity of the natural numbers. I will assume little in the way of mathematical or set theoretic knowledge for what follows, but the notation is standard in any set theory course.

Arithmetic in ZFC:

The finite ordinals are defined as:

$$0 = \emptyset$$

$$1 = \{\emptyset\} = \{0\}$$

$$2 = \big\{\emptyset, \{\emptyset\}\big\} = \{0,1\}$$

$$3 = \Big\{\emptyset, \{\emptyset\}, \big\{\emptyset, \{\emptyset\}\big\}\Big\} = \{0,1,2\} \text{ etc}\ldots$$

We get our arithmetic back from these definitions by making each set’s members unique and then taking the union of the two sets. We will observe what happens for 3 + 1.

Make 3 and 1 unique by adding a 0 or a 1 to the individual elements.

$$3 = \{(0,0),(0,1),(0,2)\} \quad ; \quad 1 = \{(1,0)\}$$

Union:

$$3 + 1 = \{(0,0),(0,1),(0,2)\} \cup \{(1,0)\} = \{(0,0),(0,1),(0,2),(1,0)\}$$

Notice that since we made every element unique, we didn't lose any information, since sets do not respect repetition (i.e. {1,1} = {1}). Now we count the number of elements, technically by typing the Mostowski collapse, to see that 3+1 has 4 elements. Thus 3 +1 = 4, as desired. We can see that, although a bit harder than counting apples, set theory provides a language to calculate innocuous arithmetic. Now I will simply state the NBG equivalent without going into the detail of the arithmetic; I will show how NBG defines the same numbers.

The Finite Ordinals in NBG:

$$0 = \emptyset$$

$$1 = \{\emptyset\} = \{0\}$$

$$2 = \left\{\{\emptyset\}\right\} = \{1\}$$

$$3 = \left\{\left\{\{\emptyset\}\right\}\right\} = \{2\}$$

Notice that in both axiom systems $0 \in 1 \in 2 \in 3$ but $0,1,2 \in 3$ is only true in ZFC arithmetic. Here we come to an important question: What exactly is the number 2? Is it defined via ZFC or via NBG or maybe some other way? Can it be all of the above? My answer, that I hope to defend, is that each number is not defined uniquely and can indeed be defined any way that preserves certain properties of the *structure* of the natural numbers. In this way, the object called the number 2 is not really the focus of mathematics, if it even exists independently of the other numbers; the number 2 gets its twoness from the structure in which it is exemplified. So already we see that the axioms provide a problem, since one axiom system cannot be favored over others except by external facts.

We notice that the empty set axiom[5], as the very first axiom, already asserts the existence of a set. Not the hypothetical existence, in its wording, but as a clear and immediate truth of sets. There is at least one set: the set with no elements. I will call these kinds of axioms *strong*. They are strong because they assert the existence of objects. One might call them "assertory". Hellman calls it *algebraic-structural*.[6] We see that in order for set theory to provide a foundation for mathematics, it must assert some object's existence. If it did not, it would not ensure that there is anything like a set, a function, a group, or even a number. So we must be able to know that mathematical objects exist, at least in a limited sense. Otherwise, we may be in the unfortunate case where we are proving things about unreal objects. While constructivism is perfectly tenable, it is not what most mathematicians believe. We will, for now, assume a realist ontology with respect to mathematics. I will defend that the objects may not exist, in the Platonic sense, but the structures in which the objects participate do exist abstractly.

Now we will look at other kinds of axiom systems. Instead of assertory axioms, these theories will mostly exemplify what I will call descriptive axioms. They don't assert the existence of any object, just the conditions under which the object would have to exist. In the terms of the language above, these would be *weak* axioms. The two examples that will suffice

[5] See Appendix A
[6] Linnebo and Pettigrew (2011) p.229

are that of category theory and group theory. We will spend much more time analyzing category theory than group theory since group theory is much less complicated when looking at the axioms. Category theory is an interesting example for an analysis of axioms. By contrasting ZFC and group theory axioms, we have a nice dyad of *strong* and *weak* axioms. Category theory, using the axioms of the Elementary Theory of the Category of Sets (ETCS) as put forward by William Lawvere[7], will provide a nice gray area for the strength of axioms.

GROUP THEORY

As the reader has likely noticed, I have put all the axioms of set theory in the Appendix because it would be confusing to include all of them when we only need 3 or 4 axioms. For group theory, however, I can list them all without risk of confusion apart from the basic mathematical language barrier.

Descriptive Axioms of Groups

*A set G with a binary operation * is a group (G,*) if:*

*G1) (Closure) For all a, b in the set G, (a*b) is also in G.*

*G2) (Associativity) For all a, b, c in the set G, [a * (b * c)] = [(a * b) * c].*

*G3) (Identity) There exists an element of the set G, say e, such that for any element b in the set G, (e * b) = (b * e) = b.*

*G4) (Inverses) For every a in the set G, there exists an element b in G such that (a*b)=(b*a) = e.*

[7] Lawvere (1964)

Not only are these easy to understand in comparison to something like the axiom of choice or infinity in set theory, they are *weak* axioms. On a close reading of these axioms, we see that the direction of fit is quite different than with the axioms of set theory. Whereas the axioms of set theory provide some implicit definition of the common notion of set through its axioms, the group axioms are the complete picture of a group. There is no dialectical approach for the group theorist, excepting the intuition we get from theorems. If we find something that fits all four group axioms, then it is a group. If we find a set, it has the ‘properties’ as outlined by the ZFC axioms. It can hardly be said that something that has no infinite sequence of membership[8] is immediately a set, for it could be vacuously true of a human that there is no infinite sequence of membership, whatever that would mean in that case. So anything that fits into the group axioms is a group *by definition* and anything that is a set fits into the set theory axioms. The direction of fit is clearly different in the two cases. This is the distinction we will make between *strong* and *weak* axioms. The *strong* axioms force the objects to exist, almost by fiat, and thus anything that is that object carries the benefit of the axioms’ self-evidence behind it. The *weak* axioms enjoy no such assumed existence. They are the object of the axioms if and only if the axioms are true for that object.

Another way to view the distinction between *strong* and *weak* is to view them as assertory and descriptive respectively. The *strong* axioms assert the existence of objects whereas the weak axioms are simply descriptive of what would have to be case for the objects to exist. In a way, the *strong* axioms are asserting unconditional truth (the empty set exists) and the weak axioms are asserting conditional truth (groups exist *if* the four axioms are fulfilled in some object).

[8] Foundation axiom in Appendix A

For groups, we have yet another definition of the natural numbers and their arithmetic. However, in group theory, since we need inverses, we have to consider **Z**, the set of positive and negative integers (and zero of course). Not only is **Z** a group, it is a field, which is very well-behaved, as mathematicians use the term. Group theory allows us to use the familiar symbol + to calculate sums as we have always done. We don't need unions or Polish lemmas or to inherit infinite ordinal arithmetic as well. We get what we learned in 1st grade. So I will not expound how we might define binary relations in group theory, as it will not illuminate anything particular to my argument. The numbers, however, may be left undefined as far as group theory is concerned. It does not matter if we denote the number 2 as ZFC or NBG does, they act the same in group theory. Here we come to a central idea that we may start to introduce mathematical structuralism. The idea that creates a reasonable base of truth in group theory is that of morphisms. Either homomorphism or isomorphism, morphisms allow statements about groups to be generalized to any group that shares the same morph (translated from the Greek morphe as form, shape, or structure). In fact, in most fields of mathematics, theorems that have wording about a unique object really are asserting uniqueness "up to some morphism". One example might be "There is a unique completion of the rationals with the usual metric into a complete space". Here, it is assumed everyone knows that the completion is only unique up to isomorphism. This is usually a trivial matter in mathematics. It is often assumed that uniqueness is "essential" such that all the objects that satisfy the unique existence are really the same. It would be akin to calling the number 2 "peanut" and the number 1 "kitten" and saying that "peanut" minus "kitten" is equal to "kitten". You really aren't changing the structure; you are changing the names. So the number 1 is the unique answer to "2-1 = x" *up to isomorphism.*

CATEGORY THEORY

Now we arrive at category theory armed with a distinction between *weak* and *strong* axioms. Which kind of axioms are in category theory (namely ETCS)? I will show that they are both. For the complete list of ETCS axioms, see Appendix B. We will only use the first two axioms in order to illustrate their dual nature.

ETCS[9]

Axiom 1: Suppose C is a category. Then C has an initial and terminal object.

For our purposes, we will denote the initial object as 0 and the terminal object as 1. I will leave the definitions of these to Appendix B, for they are not wholly important here. From this first axiom, we guess that ETCS is much like group theory. All we have to show is that C has an initial and terminal object and we have satisfied the axiom. However, this is not sufficient, given the six other axioms. The thing to notice is that in ZFC, even if you find an object that does not contradict any of the axioms it may not be a set. Similarly, categories in ETCS are not defined by their axioms in the same way groups are by theirs. The point to take away from this axiom is that 'category' is a common notion in ETCS like 'set' is in ZFC. Yet ETCS does not assert the existence of an empty category or the like, it only *defines* the necessary properties of a category by *asserting* other objects. Axiom 1 does not say that the category C has this or that initial and terminal object, just that it has them. The interesting thing about category theory is that it is not concerned with what the objects are in each category, just the roles those objects play. Technically, any object can act as the initial object and similar for the terminal object, given the right mappings.

[9] See Appendix B

So are the axioms *strong* or *weak* in ETCS? Both. They assert the existence of objects univocally but those objects are defined functionally. So the axiom asserts existence like ZFC but the constituent objects are defined like the group axioms. In many ways, I credit the autonomy of set and category theory to this dual nature of ETCS's axioms.[10] To keep this discussion parallel to all three theories analyzed, we might ask how ETCS defines the natural numbers and their arithmetic. It turns out that you can do this a number of ways, but we will look at a specific axiom that indicates an interesting trait of arithmetic in category theory. Again, I direct the reader to Appendix B for the formal definition of the terms included in the following axiom:

Axiom 8: The category of sets and mappings contains a natural-number object.

Although mathematics is a language unto itself and stands independent of our need for explication, it is often difficult to comprehend the axiom or theorem unless given some English-based summary. Linnebo and Pettigrew provide such a 'heuristic' explanation:

> In the category of sets and mappings, a *natural-number object* is a set N equipped with the two mappings $Z: 1 \to N$ and $S: N \to N$ that together guarantee the effectiveness of any recursive definition. That is, for any set X with an initial element picked out by $a: 1 \to X$ and a mapping $f: X \to X$, there is a unique $h: N \to X$ that takes the zero element of N to a and takes the successor of a 'number' in N to the element of X that results from applying f to whatever element of X was assigned to that number by h.[11]

[10] Linnebo and Pettigrew (2011) establish this autonomy sufficiently. They consider the logical, justificatory, and conceptual autonomy of the two theories.
[11] Linnebo and Pettigrew (2011) p.240-241

Although this may not help the unmathematical reader understand the axiom, it suffices to say that the natural-number object is precisely what ZFC and NBG defined in set theory, the ability to find successors. If one is familiar with the Peano axioms, then Axiom 2 provides a way to find or construct the successor function, which also provides a way to agree with Dedekind's *simply infinite system*. We also might equate it to something like the infinity axiom in ZFC. The notion of inductive set there might provide useful parallels to this natural-number object. Regardless of the convolution of this axiom, we see that category theory does indeed provide a model for arithmetic, even if it doesn't look anything like what we might be used to.

So we come back to the problem hinted after the discussion on ZFC versus NBG: What is a number? For the argument here, we restrict ourselves to the intuitive finite numbers like 4 and 100. Each axiomatic theory we looked at gave a different answer. They all express arithmetic and numbers in the language of the theory and most of them are mutually exclusive definitions. The number 2 cannot be both $\{0,1\}$ and $\{1\}$ since by any definition of equality of sets (i.e. either extensionality axioms in ZFC or NBG), $\{0,1\} \neq \{1\}$. The question then might be what axiomatization we ought to prefer. Which system accurately represents what the number 2 is, if any? All of the theories I have so far explained represent the natural numbers (or integers in the case of group theory) accurately. In what follows I will defend the position in the philosophy of mathematics that is called structuralism. But before that, I will note that just because all the representations of numbers are accurate in the theories thus far, I do not mean that they are equal representations. I mean something much weaker when I say they all represent the numbers accurately: they all exemplify the system of natural numbers, independently of how they represent the individual number-objects.

Defending Structuralism from the Axioms:

As the name suggests, structuralism is the philosophical stance that mathematics isn't about objects per se, but the structures that the objects are in. That is to say that when we add 2 and 3, we are operating on object stand-ins. Realists that are not structuralists are committed to the notion that $\{0,1\} \neq \{1\}$ implies the number 2 in ZFC is a completely different number than the number 2 in NBG. They are not simply different in definition, but ontological status. These Platonists are beholden to the belief that essential uniqueness[12] is not uniqueness at all. The structuralist, on the other hand can embrace the view that no matter how you define two, if it is the third number in any natural number *structure*, then it *is* two (two is the third number since we will be using the convention that 0 is the first number).

The theory of groups provides a perfect example of such a structure. We care not what the objects really are, and we usually denote them arbitrarily, but we do care about what the structure is like. If, say, the structure of a group is cyclic and of finite order, then it is the same, structurally, to $\mathbf{Z_n}$. In group theory, they are not treated just conveniently the same, they are quite importantly the same. Here, Hellman's use of the term *structural-algebraic*[13] will provide some insight into how structuralists think of group morphisms, noting that group theory is algebraic. Structuralism can be viewed similarly to functionalism in other areas of philosophy. It is not about what the object is, but what role the object plays. Category theory makes this belief explicit.

[12] See bottom of page 6.
[13] Linnebo and Pettigrew (2011) p.229

In set theory sets are defined for the most part using extension. In category theory, namely ETCS, categories and what are roughly the same as sets, are defined much more intensionally. Gödel and others had a notion of set that might be called iterative[14]. After a fashion, this iterative notion helped justify some of the axioms in ZFC. After all, that is what Gödel was attempting to do, along with Boolos and Parsons. We should not fall into the obvious objection that axioms are self-evident or immediate and should need no justification. This is not precisely what the iterative concept of set did. It provided incentive for some of the *strong* axioms like the empty set axiom. Under the iterative conception, sets contained objects of strictly lower levels of iteration. The set {1,2,3,4} is of level 1 since it contains simple (atomic) objects. Here just assume 1,2,3,4 are objects and not structural positions. If a set contained other sets, then the universal set would be of a higher level in the iterative hierarchy than the member sets. At the very least, we see that in this conception, there 'must' be a set with level 0, that is, a set with no atomic objects or any higher order objects. If it helps, this can be viewed as analogous to type theory. So we have some motivation for the empty set, the power set, the foundation axiom and so on. Thus in sets, we must know what every member looks like to really claim the set exists, unless guaranteed by the axioms.

For categories, though, we aren't bound by the proper sets of set theory; we have proper classes and others as well. Thus our definitions of classes and categories are much less specific and more functional. I don't care what the initial object is, just that it exists. Since set theory can be expressed as a sub-theory of category theory, the *strong* axioms of set theory can be expressed in terms of the dually *strong* and *weak* axioms of ETCS. This implies that with category theory, we avoid commitment to Platonism from the existence of one empty set; instead we are

[14] Gödel (1964)

committed to the idea that there is something that is structurally the same as the empty set in categories. Since this is not wholly *strong*, we are left with the much more attractive stance of structuralism. And if one believes Linnebo and Pettigrew sufficiently established that ETCS is an autonomous foundation for mathematics (but can still express set theory), then we have a foundation for mathematics in structuralism that fits our parallel philosophy. We recognize the mathematician's focus on morphisms and the epistemological problems of particular object Platonism and see that structuralism fits better with our mathematician and philosophical desiderata.

The axioms of ZFC are not necessarily *strong*, then. Since ETCS expresses the same truths while still maintaining descriptive, definitive axioms, the *strong* axioms are superfluous to an ontology that aims only to reflect mathematical objects. So we have arrived at the position that the only axioms that need be supposed are those that include descriptive or *weak* axioms. As group theory exemplified, when we have *weak* axioms, we have structurally focused theories. We conclude that every mathematical field expressible via category theory is structure-based and thus structuralism most accurately characterizes mathematics in accord with our parallel philosophy approach.

Realism from the Empirical:

In his 1960 essay, Eugene Wigner characterizes mathematics in the physical sciences as "unreasonably effective"[15]. He argues that mathematics is so effective and expressive in science that it surpasses even applied mathematics' ability. We know that physicists constantly use math in determining friction, thermodynamics, and ion flow, but this might only be the same sense in

[15] Wigner (1960)

which a cashier uses math. True, it is math, but that might just be a convenient short-hand for physical facts. Applied mathematics does not imply realism at all; it might even support something like constructivism. However, Wigner shows us that pure mathematics has a much more important and "sovereign" role in explaining physical theories. I will use two examples in physics that illustrate pure mathematics unreasonable effectiveness. The first is that of Einstein's relativity. Until quite recently (in the history of mathematics), geometry was rather stationary and as close to completion as we might expect a mathematical field to be. Not until Bolyai-Lobachevskian geometry did mathematicians' have insight into non-Euclidean geometry. Einstein used this new geometry to its fullest extent with his two relativistic theories. If we are to suppose his theories are accurate representations of the physical universe, then we might suppose that his theories are in terms of physics. However, his theories are stated almost entirely in pure mathematics. It is this fact, that entire theories are expressed in mathematics, which supports Wigner's unreasonable effectiveness.

The second, more controversial theory is that of super-string theory, or M-theory. The question of its scientific feasibility is not considered here. After all, M-theory may not really be falsifiable. But ignoring that debate, we see that M-theory is one of the only candidates for the theory of everything. Its predictions and consequences were derived completely from pure mathematics, as theoretical physics often is. So we begin to question why mathematics is so effective in the natural world that it may describe all of existence via M-theory.

Now we deal with how these questions deal a direct blow to non-realism. If we accept the existence of the physical world, we usually suppose it is independent of the mind. So if something is sufficiently tied to that independent world, we suspect it too is independent of the existence of minds. M-theory, being a theory of everything, and Einstein's relativity, being one of the most successful theories of all time, provide strong argument for the tie between the physical and the mathematical. It is difficult to see how human invention can provide such beautiful theoretical models when they are only created from mathematics, not observation. If the theories were derived or expressed in terms of observation, we might suppose some invention possible. However, M-theory, at least, shows us that pure mathematics can arrive at a theory of everything beyond observation. And we should also see that M-theory was not arrived at accidentally, that it was not coincidence that it describes the universe.

So if we hold anything but even the weakest realism, we must explain how mathematics can explain, express, predict, and incorporate the entirety of existence *especially that which is beyond current observational ability*. Realism surely is the most apt stance to address this: mathematics is true of the world independent of us because mathematics itself is independent of us. But realism is not my entire agenda.

Extending this notion of unreasonable effectiveness to structuralism will take less sophisticated theories. Take the arithmetic we learn at a very young age. We saw earlier that $3 + 1 = 4$ can be verified by ZFC, NBG, group theory, and category theory. But they disagreed about what 3, 1, and 4 are, ignoring the relations + and =. Now observe that if I have one apple and add it to a group of 3 apples, I will have four apples. This is usually how we learn simple math. But the arithmetic works whether we have apples, planets, atoms, volume, sets, money, etc. Why should this abstraction of addition be so accurate that we say it is true regardless of the objects? The answer is structuralism. It doesn't matter which objects are operated upon; it only matters in what structure those operations are defined. For all of the above examples, the structure of the finite set provides the basis for a common arithmetic. Any way you define the objects of the finite set will suffice as long as you maintain the *structure* of arithmetic. This is why mathematics is so appropriate to the physical world. I don't care if it's a photon or a rocket that is travelling at relativistic speeds, Einstein's theory will describe the time dilation. Similarly, the mathematician doesn't care if $2 = \{0,1\}$ or $2 = \{1\}$, as long as it operates in the same structure as we expect the numbers to.

Finally we see that structuralism describes both mathematics and its relation to the physical world. Structuralism in mathematics, therefore, supports the physicist's use of mathematics (pure and applied) and the mathematician's research while also maintaining realism and ontological parsimony. I leave the epistemological questions for a later paper, but I refer the reader to Shapiro's important structuralist work *Philosophy of Mathematics: Structure and Ontology* for a discussion on epistemology. The same book will also provide a helpful discussion on *ante rem* and *an re* structuralism. This distinction is an answer to the question "What ontological status is ascribed to the places in the structures?" When I put an apple in the structure of finite arithmetic, am I replacing the first-order location of the structure with a physical object, or is the office of the number 2 an object in itself? These questions are important but separate from the current investigation.

Conclusion:

We see that structuralism should be favored to other positions in the philosophy of mathematics. Non-realist positions cannot answer the question of unreasonable effectiveness and object Platonists cannot resolve the disparity between ZFC and NBG when defining the finite ordinals. Therefore we are committed to a structural realism. For a more sophisticated discussion on the disparity between ZFC and NBG and similar problems for object Platonists, Benacerraf's Dilemma serves well.[16]

[16] Benacerraf (1973)

Appendix A

ZFC[17]

Empty Set Axiom

$\exists! A \ \forall x \ (x \notin A)$

Extensionality Axiom

$\forall A \ \forall B \ [\forall x \ (x \in A \Leftrightarrow x \in B) \Rightarrow (A = B)]$

Pairing Axiom

$\forall x \ \forall y \ \exists! A \ \forall z \ [z \in A \ \Leftrightarrow \ (z = x \ or \ z = y)]$

Union Axiom

$\forall F \ \exists! A \ \forall x \ [x \in A \ \Leftrightarrow \ \exists Y \in F \ (x \in Y)]$

Power Set Axiom

$\forall A \ \exists! F \ \forall X \ (X \in F \ \Leftrightarrow \ X \subseteq A)$

Comprehension Scheme

$\forall A \ \exists! B \ \forall x \left[x \in B \Leftrightarrow \left(x \in A \ and \ P(x)\right)\right]$ where P(x) is understood as a "property".

Infinity Axiom

$\exists I \ [\emptyset \in I \ and \ \forall x \ (x \in I \Rightarrow x \ \cup \{x\} \in I)]$

Replacement Scheme

$\forall A \ \left[\left(\forall x \in A \ \exists y \ P(x, y)\right) \Rightarrow \left(\exists B \ \forall x \in A \ \exists y \in B \ P(x, y)\right)\right]$

Foundation Axiom

$\forall S \ (S \neq \emptyset \ \Rightarrow \exists x \in S \ \forall y \in S \ (y \notin x))$

Axiom of Choice

$\forall F \ \exists \text{function } c \ \forall A \in F \ (A \neq \emptyset \Rightarrow c(A) \in A)$

[17] Schlimmerling (2011)

Appendix B

ETCS[18]

Definition: Suppose C is a category and 0 and 1 are objects of C. Then

(i) 0 is an **initial object** of C if, for every A, there is a unique arrow 0_A: 0→A.
(ii) 1 is a **terminal object** of C if, for every A, there is a unique arrow 1_A: A→1.

<u>Axiom 1</u>

There is an initial object and a terminal object.

<u>Axiom 2</u>

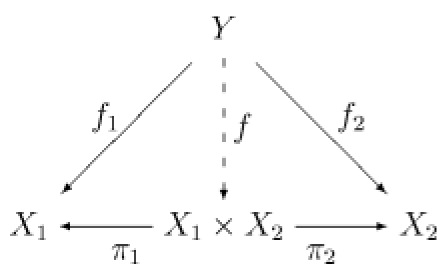
0 and 1 are not isomorphic.

Definition: Suppose Y is a category and X_1 and X_2 are objects of Y.

Then π_1, π_2: $X_1 \times X_2 \Rightarrow X_1, X_2$ is the **Cartesian product** of X_1 and X_2 if, for any

f_1, f_2: Y $\Rightarrow$ X_1, X_2, there is a unique f: Y $\Rightarrow X_1 \times X_2$ such that the following diagram

commutes:



<u>Axiom 3</u>

For any two sets X_1 and X_2, there is a Cartesian product π_1, π_2: $X_1 \times X_2 \Rightarrow X_1, X_2$.

Definition: Suppose C is a category and f,g: A$\Rightarrow$B are arrows in C. Then e: E →A is an **equalizer** for f and g if

(i) $fe = ge$
(ii) If e\`: E\`→ A and $fe\` = ge\`$, then there is a unique k: E\` → E such that $e\` = ek$.

<u>Axiom 4</u>

For any two mappings f,g: A $\Rightarrow$ B, there is an equalizer e: E → A

[18] Linnebo and Pettigrew (2011)

Axiom 5 [19]

(i) There is a subobject classifier *true*: $1 \rightarrow \Omega$.
(ii) For any set A, there is a power object P(A) equipped with $\in_A$: $A \times P(A) \rightarrow \Omega$.

Axiom 6

If f,g: $A \Rightarrow B$ and $fx = gx$ for all x: $1 \rightarrow A$, then $f = g$.

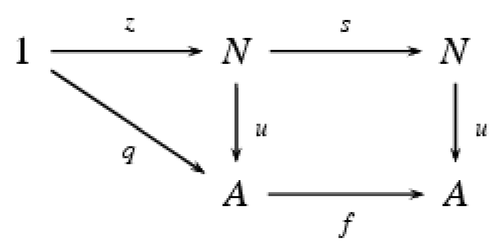
Definition: Suppose C is a category and j: $A \rightarrow B$ is an arrow in C. Then j is **epic** if, for any C and any two distinct arrows g, h: $B \Rightarrow C$, the arrows gj and hj are also distinct.

Axiom 7

If j: $A \rightarrow B$ is epic, then there is g: $B \rightarrow A$ for which $jg = \mathrm{Id}_B$.

Definition: Suppose C is a category with terminal object 1. Then N together with

z: $1 \rightarrow N$ and s: $N \rightarrow N$ is a **natural-number object** in C if, for any object A and arrows q: $1 \rightarrow A$ and f: $A \rightarrow A$, there is a unique arrow u: $N \rightarrow A$ such that the following diagram commutes:



Axiom 8

The category of sets and mappings contains a natural-number object.

[19] I omit the definitions of subobject classifier and power object because it is much easier to understand them in terms of the ZFC axioms. A subobject classifier is basically a function that is equivalent to the characteristic function, where you can pick out a subset of any set. The power object is much like the power set. Since we can find 'subsets' with the subobject classifier, we can put all the 'subsets' into another object, namely the power object. This is as close as ETCS gets to having subsets, power sets, and local membership.

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